

# Finite Time Adaptive Optimal Integral Sliding Mode Control for a Class of Uncertain Second Order Nonlinear Systems with Input Nonlinearity

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## Abstract

*In this paper, a new robust controller based on geometric homogeneity and adaptive integral sliding mode is proposed for a class of second order systems. The upper bound of the system disturbances is not required. Fully unknown parameters have been considered in the described model and its finite-time convergence to zero equilibrium point is proved. Moreover, the controller is developed in the presence of control singularity and unknown non-symmetric input saturation. The finite time stability of the proposed controller has been proved via classical Lyapunov criteria. In order to tune the control parameters, all the positive constant gains are optimized by ant colony optimization algorithm during the offline input-output training data. Two polar robots are introduced to show the performance of the designed controller. The robustness and error accuracy are proved in simulation results. Moreover, the effects of input nonlinearity such as input saturation have been considered in the simulation.*

**Keywords:** Adaptive control, Integral sliding mode control, Input saturation, Ant colony optimization

## 1. Introduction

Sliding mode control (SMC) is known to be an efficient and powerful control technique applicable to a wide class of nonlinear systems. The SMC methods have brought many advantages, such as insensitivity to model uncertainty, external disturbances and unknown parameters and it also shows the fast response. The control design of uncertain systems is still challenge and it has drawn serious attention from the research community. Among the existing control techniques [1-2] variable structure control [3] and sliding mode control are powerful control schemes against perturbation. The first step in the sliding mode control design is to choose a suitable manifold such which the state variables restrict to the manifold with a desired dynamic and converge to their equilibrium. By using SMC, the states of the system are forced to move along the chosen manifold in the state space, called the sliding surface. The next step is to defining an input signal to ensure that the error system trajectories will reach to the sliding surface and stay on it forever. After reaching the sliding manifold, the system becomes totally insensitive to parametric uncertainties and external disturbances.

However, conventional switching manifolds are usually linear hyperplanes which guaranty the asymptotic stability of dynamic errors; but, dynamic error cannot converge to zero in finite time. The sliding mode parameters can be adjusted to get faster error

convergence, however, this will increase the control gain, which may cause much more chattering on the sliding surface and it deteriorates the system performance. Different from the traditional control strategies where the tracking errors are required to converge in an asymptotic way, the terminal sliding mode (TSM) control achieves finite-time convergence of the tracking errors [4-5]. With this ability, the closed loop system can be accurately and efficiently controlled to the given command. Accordingly, many application examples of TSMC have been considered to provide fast and finite time control performance as well as high precision [6-8]. In [9], the fast terminal dynamics was proposed and used in the design of the sliding-mode control for SISO nonlinear dynamical systems. The authors of [10] proposed nonsingular terminal sliding mode called NTSMC to solve nonsingular problem. Many other researchers have worked on finite time sliding mode control [10-11]. A geometric homogeneity based finite time controller is designed for a chain of integrator system in [12-13]. A higher order sliding mode control based on geometric homogeneity is also developed in [14]. Adaptive integral high order sliding mode control considering uncertainty based on a geometric homogeneity feedback is proposed in [15]. Moreover, fractional order controller is designed for flexible robots in [16]. Fractional order sliding mode is also presented in [17] for nonlinear systems.

Adaptive sliding mode control strategy strengthens the control system's robustness against modeling uncertainties and external disturbances and decreasing signal control amplitude. Adaptive laws can provide suitable estimates for the uncertainties and unknown parameters. In order to provide finite time adaptive sliding mode, Plestan [18] have improved the adaptive sliding mode control. Furthermore, twisting algorithm and sliding mode are used to observer non measurable information for nonlinear systems [19].

The control input saturation is one of the most important nonlinearities which should be explicitly considered in the control design. Analysis and design of control systems with input saturation have been studied in [20-23]. Globally stable adaptive control was presented for minimum phase SISO plants with input saturation [24]. Moreover, sliding control scheme has been presented in the presence of input saturation [25]. Thus, the robust tracking control needs to be further developed for the uncertain nonlinear system with control singularity case and unknown non-symmetric input saturation. In this paper, the proposed sliding mode control will be considered with control singularity and unknown non-symmetric input saturation.

ACO algorithm has been described in [26]. The main idea of ACO is to model a problem as the search for a minimum cost path in a graph. Artificial ants as if walk on this graph, looking for cheaper paths. Each ant has a rather simple behavior capable of finding relatively costlier paths. Cheaper paths are found as the emergent result of the global cooperation among ants in the colony. In this paper, ACO will be used to train the controller gains for an optimum solution according to their cost function. Therefore, after each search, the gains are updated in appropriate range and finally among all the found solutions, the best one will be chosen to be used in this controller.

In this paper, an integral Non-singular Terminal ASMC (OITASMC) has been designed for second order systems. In this method, adaptive laws are used to estimate the unknown parameters and higher bounds of disturbances. Moreover, optimal learning will be used to search an optimal solution for constant gains of the controller. The main contribution of this work is to redesign a SMC controller that is robust against unknown parameters and other perturbations of second order systems. Integral state will be

considered as system states in the sliding surface design. In order to achieve a free steady state error and accurate tracking response in a finite time, it is designed based on geometric homogeneity. The singularity of unknown non-symmetric input saturation will be considered in the controller design. Moreover, ACO algorithm will be applied for the offline optimization. The main advantages of this work are as follows:

- Zero steady-state errors will be acquired according to using integral action in non-singular terminal sliding mode design.
- Finite time robust tracking and stabilization will be guaranteed through the Lyapunov criteria.
- Optimal solution will be found to minimize the quadratic cost function.

The organization of this paper is as follows: the next section details some preliminary lemmas and assumptions with the problem formulation. The proposed control strategy is described in the third section. The control input is developed considering control singularity and input saturation in this section. This section concerns offline ant colony optimization for best possible gain selection. Case study is introduced in forth section. Simulation results are brought and discussed in the 5<sup>th</sup> section. Finally, the major conclusions of the work are drawn.

## 2. Preliminary lemmas and problem formulation

In this section, the system description and tracking problem are explained and some necessary lemmas are introduced. Let the nonlinear second order system with parametric uncertainties and environmental disturbances as follows:

$$\begin{aligned} \dot{x}_{2k-1}(t) &= x_{2k}(t) \\ \dot{x}_{2k}(t) &= f_k(x, t) + \Delta f_k(x) + F_k(X)\alpha + g_k(x)u_k(t) + d_k(t) \end{aligned} \quad (1)$$

where  $x(t) = [x_1(t), x_2(t), \dots, x_{2n}(t)]^T$  is the state vector for  $n=1,2,\dots$  and  $t \in \mathbb{R}$ . It is assumed that  $g_k(x) \neq 0$  and  $g_k^{-1}(x) \neq 0$  for the first controller design.  $d_k(t)$  is an external time-varying disturbance and  $\Delta f_k(x)$  is a bounded unknown uncertainty based on assumption which will be proposed later in this section. It is clear that lots of systems could be represented by Eq. (1) and it usually can be considered as mechanical systems like robot manipulators. Moreover, many chaotic systems are known with the same dynamic as Eq. (1) [27].

**Lemma1.** [28] For  $x_i \in \mathbb{R}$ ,  $i=1,2,\dots,n$ ,  $0 < p \leq 1$  is a real number, then the following inequality holds:

$$\left(|x_1| + |x_2| + \dots + |x_n|\right)^p \leq |x_1|^p + |x_2|^p + \dots + |x_n|^p \quad (2)$$

**Lemma2.** [12] Assume that a continuous, positive definite function  $V(t)$  satisfies the following differential inequality:

$$\dot{V}(t) \leq -cV^\varsigma(t) \quad \forall t \leq t_0 \quad V(t_0) \geq 0 \quad (3)$$

where  $c > 0$ ,  $0 < \zeta < 1$  are two constants. Then, for any given  $t_0$ ,  $V(t)$  satisfies the following inequality:

$$V^{1-\zeta}(t) \leq V^{1-\zeta}(t_0) - c(1-\zeta)(t-t_0), \quad t_0 \leq t < t_1 \quad (4)$$

and  $V(t) \equiv 0$ ,  $\forall t \geq t_1$  with  $t_1$  given by

$$t_1 = t_0 + \frac{V^{1-\zeta}(t_0)}{c(1-\zeta)} \quad (5)$$

**Lemma 3.** [29] Let  $k_1, k_2, \dots, k_n > 0$  be such that the polynomial  $s^n + k_n s^{n-1} + \dots + k_2 s + k_1$  is Hurwitz, and consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u \end{aligned} \quad (6)$$

There exists  $\varepsilon \in (0, 1)$  such that, for every  $\beta \in (1 - \varepsilon, 1)$ , the origin is a globally finite-time-stable equilibrium for the system (6) under the feedback

$$u = -k_1 |x_1|^\alpha \text{sign}(x_1) - \dots - k_n |x_n|^\alpha \text{sign}(x_n) \quad (7)$$

where  $\alpha_1, \dots, \alpha_n$  satisfy  $\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}$ ,  $i = 2, \dots, n$ , with  $\alpha_{n+1} = 1$  and  $\alpha_n = \beta$ .

**Assumption 1.** The environmental disturbance  $d_k(t)$  and uncertainty term  $\Delta f_k(x)$  are assumed to be bounded with the following inequality

$$|\Delta f(x)| \leq \rho_1, |d(t)| \leq \rho_2 \quad (8)$$

$$|\Delta f_k(x)| + |d_k(t)| \leq W_k \quad (9)$$

where  $W_k$  is a given positive constant for any real number  $k$ .

**Assumption 2.** The reference signal  $r(t) \in \mathfrak{R}$ , and its  $n$  order time derivatives  $r^n(t)$  are bounded and piecewise continuous signals.

**Assumption 3.** It is assumed that uncertainties of the system are bounded and they are also governed by a nonnegative smooth function with an unknown constant. There exists the term

$$F(x)\alpha$$

where  $\alpha$  is a known positive constant, and  $F(x)$  is the  $m \times n$  matrix whose elements are continuous nonlinear functions,  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_m]^T$  is the  $m \times 1$  unknown parameter vector of the system, and  $m$  is the number of uncertainties. The  $k^{\text{th}}$  row of the matrix  $F(x)$  is

$$F(x)\alpha. \quad (10)$$

**Assumption 4.** The unknown vector parameters  $\alpha$  are norm bounded.

$$|\alpha| \leq D \quad (11)$$

### 3. The controller design

#### 1.3 Controller design without considering input saturation

The classical state feedback controller makes it impossible to reject the effects of disturbances, particularly input disturbances. One of the very useful methods of disturbance rejection is adding an integral state of error to the feedback controller to ensure a unitary static closed-loop gain, i.e., the gain between the reference command and the output associated to the reference. Also, it is used to eliminate steady state errors and ensure the tracking accuracy.

In this section, OITASMC will be designed for second order systems with and without input nonlinearity. Before that, we briefly survey similar researches. In [26], PID controller has been designed with optimal gains, but this method is not robust against uncertainty and disturbance and just can improve the classical PID performance. A linear ASMC has been designed in [27], but it cannot ensure the finite time and robust stability. Finite time approach has been used in [28] for the problem of synchronization, but like [30] the upper bound of uncertainties and disturbance are necessary to be known. Also, integral action has not been considered. Finite time theory and feedback state have been introduced in [29], but this strategy is not appropriate for uncertain systems.

In order to solve the tracking command problem, the error between states and commands with integral state are defined as below:

$$\begin{aligned}\dot{\xi}_{2k-1} &= x_{2k-1} - x_{2k-1,r} \\ e_{2k-1} &= x_{2k-1} - x_{2k-1,r} \\ e_{2k} &= x_{2k} - \dot{x}_{2k-1,r}\end{aligned}\quad (12)$$

where  $\dot{x}_{i,r}$  is the reference signal for the  $i^{\text{th}}$  state.

It should be noted that an integral term has been added to state errors. This term makes it possible to achieve the zero steady-state error. Therefore, substituting Eq. (1), the error dynamic could be obtained as follows:

$$\begin{aligned}\dot{\xi}_{2k-1} &= e_{2k-1} \\ \dot{e}_{2k-1} &= e_{2k} \\ \dot{e}_{2k} &= f_k(x, t) + \Delta f_k(x) + F_k(X)\alpha + g_k(x)u_k(t) + d_k(t) - \ddot{x}_{2k-1,r}\end{aligned}\quad (13)$$

In order to design the sliding mode controller, an integral sliding manifold has been proposed to guarantee the system convergence to zero in finite time. In this paper, a novel non-singular PID sliding surface is introduced as:

$$\begin{aligned}s_k &= k_k^i \int_0^t |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) d\tau + k_k^p \int_0^t |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) d\tau \\ &\quad + k_k^d \int_0^t |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) d\tau + \dot{e}_{2k-1} - e_2(0)\end{aligned}\quad (14)$$

where  $c = 1, \dots, n/2$ ,  $0 < \alpha_i < 1$  for  $i = 1, 2, 3$  and they are calculated as  $\alpha_1 = \frac{\beta}{3 - 2\beta}$ ,

$\alpha_2 = \frac{\beta}{2 - \beta}$ ,  $\alpha_3 = \beta$ , where  $0 < \beta < 1$  and  $k_k^p$ ,  $k_k^i$  and  $k_k^d$  are positive constants such

that the polynomial  $s^3 + k_k^d s^2 + k_k^p s + k_k^i$  becomes Hurwitz according to the Lemma (3).

According to the Lemma (3), it is obvious that if the finite time stability of the sliding surface is proved, the error trajectory will converge to zero for any initial state of system. It can be concluded that the system is stable and all of the error states converge to zero in finite time.

When the system operates in the sliding surface, the following equations are satisfied:

$$s_k(t) = 0 \quad (15)$$

$$\dot{s}_k(t) = 0. \quad (16)$$

The first order derivative of the sliding surface is obtained as:

$$\begin{aligned} \dot{s}_k = & k_k^i |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) + |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) \\ & + |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) + \ddot{e}_{2k} \end{aligned} \quad (17)$$

Also, it can be rewritten as:

$$\begin{aligned} \dot{s}_k = & k_k^i |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) + |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) \\ & + |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) + \ddot{e}_{2k-1} \end{aligned} \quad (18)$$

From the above equation, we get:

$$\begin{aligned} \dot{e}_{2k} = & -k_k^i |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) - |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) \\ & - |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) \end{aligned} \quad (19)$$

Consider the system error equations (13), the close-loop system can be rewritten as:

$$\begin{aligned} \dot{\xi}_{2k-1} &= e_{2k-1} \\ \dot{e}_{2k-1} &= e_{2k} \\ \dot{e}_{2k} &= -k_k^i |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) - |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) \\ & - |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) \end{aligned} \quad (20)$$

Consider the proposed integral non-singular terminal sliding mode dynamics with Eq. (14). From Lemma (3), the sliding surface is a globally finite-time stable equilibrium for the system with Eq. (1). The reaching time for the convergence of the sliding surface is not estimated here, but we assume that there exist a finite time  $T_1$  that system states converge to zero (their equilibrium) in this manifold.

After establishing integral sliding manifold based on geometric homogeneity, a control law should be designed to force error trajectories to reach on the sliding surface within a finite time and remain on it forever, in the presence of bounded external disturbances, uncertainties and unknown parameters. In order to design a robust controller in the presence of bounded external disturbances  $d_k$  and uncertainties  $\Delta f_k(x)$  should be determined precisely and unknown parameters should be estimated to be employed in control law. However in practice, they are fully unknown. In order to improve the robustness of the controller, the switching control is proposed as follows:

$$u_k(t) = g_k^{-1}(x) \left( \begin{array}{l} -k_k^j |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) - |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) - |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) \\ -f_k(x,t) - F_k(X)\hat{\alpha} - \hat{k}_k \text{sign}(s_k) - \frac{s_k}{\|s\|^2} (\|\hat{\alpha}\| + D) \end{array} \right) \quad (21)$$

Defining the adaptation errors as  $\tilde{\alpha} = \hat{\alpha} - \alpha$  and  $\tilde{k} = \hat{k} - k$ , the parameters  $\hat{k}$  and  $\hat{\alpha}$  will be estimated using the following adaptation laws:

$$\dot{\hat{\alpha}} = [F(x)]^T \mu \quad (22)$$

$$\mu = [s_1, s_2, \dots, s_k]^T \quad (23)$$

$$S(t) = [s_1, s_2, \dots, s_k]^T \quad (24)$$

$$\dot{\hat{k}}_k = \theta |s_k| \quad (25)$$

**Theorem1.** For the error dynamics (13) with the control law (24), the states of the system are finite time stable and will be converged to the sliding surface  $s(t) = 0$  in a finite time as follows:

$$T_2 \leq \frac{2 \left( \sum_{k=1}^n \left[ \frac{1}{2} s_k(0)^2 + \frac{1}{2} (\hat{k}_k(0) - k_k)^2 \right] + \frac{1}{2} \|\hat{\alpha}(0) - \alpha\|^2 \right)^{1/2}}{\sqrt{2\omega_k}} \quad (26)$$

where  $\omega_k$  is non zero term.

**Proof.** Let choosing a positive definite function in the form of:

$$V(t) = \frac{1}{2} \sum_{k=1}^n \left[ s_k^2 + \gamma_k (\tilde{k}_k)^2 \right] + \|\tilde{\alpha}\|^2 \quad (27)$$

The derivative of  $V(t)$  could be obtained as:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \dot{s}_k(t) + \gamma_k \tilde{k}_k \dot{\tilde{k}}_k \right] + \tilde{\alpha}^T \dot{\tilde{\alpha}} \quad (28)$$

Substituting Eq. (17) into the above equation, we get:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( \begin{array}{l} k_k^j |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) + |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) \\ + |\dot{e}_{2k-1}(\tau)|^{\alpha_3} \text{sign}(\dot{e}_{2k-1}) + \dot{e}_{2k} \end{array} \right) + \gamma_k \tilde{k}_k \dot{\tilde{k}}_k \right] + \tilde{\alpha}^T \dot{\tilde{\alpha}} \quad (29)$$

From  $\dot{e}_{2k}(t)$  with Eq. (19), we get:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( \begin{array}{l} k_k^j |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) + |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) + |e_{2k}(\tau)|^{\alpha_3} \text{sign}(e_{2k}) \\ + f_k(x,t) + \Delta f_k(x) + F_k(X)\alpha + g_k(x)u_k(t) + d_k(t) - \ddot{x}_{2k-1} \end{array} \right) + \gamma_k \tilde{k}_k \dot{\tilde{k}}_k \right] + \tilde{\alpha}^T \dot{\tilde{\alpha}} \quad (30)$$

Substituting  $u_k(t)$  from Eq. (21) into the above equation yields:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( \begin{array}{l} k_k^j |\xi_{2k-1}(\tau)|^{\alpha_1} \text{sign}(\xi_{2k-1}) + |e_{2k-1}(\tau)|^{\alpha_2} \text{sign}(e_{2k-1}) + |e_{2k}(\tau)|^{\alpha_3} \text{sign}(e_{2k}) \\ + \Delta f_k(x) + F_k(X)\alpha - F_k(X)\hat{\alpha} - \hat{k}_k \text{sign}(s_k) - \frac{s_k}{\|s\|^2} (\|\hat{\alpha}\| + D) + d_k(t) \end{array} \right) + \gamma_k \tilde{k}_k \dot{\tilde{k}}_k \right] + \tilde{\alpha}^T \dot{\tilde{\alpha}} \quad (31)$$

According to the adaption laws Eq. (25) and using assumptions (2) and (3), we get:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( W_k(t) + F_k(X)\alpha - F_k(X)\hat{\alpha} - \hat{k}_k \text{sign}(s_k) - \frac{s_k}{\|s\|^2} (\|\hat{\alpha}\| + D) \right) + \gamma_k \tilde{k}_k \theta |s_k| \right] + \tilde{\alpha}^T [F(x)]^T \mu \quad (32)$$

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( W_k(t) - F_k(X)\tilde{\alpha} - \hat{k}_k \text{sign}(s_k) - \frac{s_k}{\|s\|^2} (\|\hat{\alpha}\| + D) \right) + \gamma_k \tilde{k}_k \theta |s_k| \right] + \tilde{\alpha}^T [F(x)]^T \mu \quad (33)$$

by adding and removing  $k_k \text{sign}(s_k)$  we obtain:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( W_k(t) - F_k(X)\tilde{\alpha} - \hat{k}_k \text{sign}(s_k) - k_k \text{sign}(s_k) + k_k \text{sign}(s_k) - \frac{s_k}{\|s\|^2} (\|\hat{\alpha}\| + D) \right) + \gamma_k \tilde{k}_k \theta |s_k| \right] + \tilde{\alpha}^T [F(x)]^T \mu \quad (34)$$

Since  $\sum_{k=1}^n s_k(t) F_k(x) \tilde{\alpha} = \tilde{\alpha}^T [F(x)]^T \mu$  and  $\sum_{k=1}^n s_k \frac{s_k}{\|s\|^2} = 1$ , we get:

$$\dot{V}(t) = \sum_{k=1}^n \left[ s_k(t) \left( W_k(t) - F_k(X)\tilde{\alpha} - \hat{k}_k \text{sign}(s_k) - k_k \text{sign}(s_k) + k_k \text{sign}(s_k) - \frac{s_k}{\|s\|^2} (\|\hat{\alpha}\| + D) \right) + \gamma_k \tilde{k}_k \theta |s_k| \right] - (\|\hat{\alpha}\| + D) \quad (35)$$

then, the above equation changes to the below inequality:

$$\dot{V}(t) = \sum_{k=1}^n \left[ |s_k| W_k(t) - \tilde{k}_k |s_k| - k_k |s_k| + \gamma_k \tilde{k}_k \theta |s_k| \right] - (\|\hat{\alpha}\| + D) \quad (36)$$

$$\dot{V}(t) = \sum_{k=1}^n \left[ |s_k| (|W_k(t) - k_k| - \tilde{k}_k (|s_k| - \gamma_k |s_k| \theta)) \right] - (\|\hat{\alpha}\| + D) \quad (37)$$

Under Assumption (4) and since  $\|\hat{\alpha} - \alpha\| \leq \|\hat{\alpha}\| + \|\alpha\| \leq \|\hat{\alpha}\| + D$ , it can be concluded that  $-(\|\hat{\alpha}\| + D) \leq -\|\hat{\alpha} - \alpha\|$ , which leads to:

$$\dot{V}(t) = \sum_{k=1}^n \left[ -|s_k| (-|W_k(t) + k_k| - \tilde{k}_k (|s_k| - \gamma_k |s_k| \theta)) \right] - \|\hat{\alpha} - \alpha\| \quad (38)$$

Since there is always a  $k_k > 0$  such that  $\tilde{k}_k > 0$  for all  $t > 0$ , we can conclude that:

$$\dot{V}(t) = \sum_{k=1}^n \left[ -\lambda_k |s_k| - \Omega_k \tilde{k}_k \right] - \|\hat{\alpha} - \alpha\| \quad (39)$$

where  $\lambda_k = -|W_k(t) + k_k|$  and  $\Omega_k = |s_k| - \gamma_k |s_k| \theta$  for  $k = 1, 3, \dots, n$  as  $n$  is a real number.

So, we get:

$$\dot{V}(t) \leq -\sum_{k=1}^n \left[ \lambda_k |s_k| + \Omega_k \tilde{k}_k \right] - \|\hat{\alpha} - \alpha\| \quad (40)$$

From Lemma (1), it can be concluded that:

$$\dot{V}(t) \leq -\sqrt{2}\omega_k \left( \sum_{k=1}^n \left[ \frac{1}{2} s_k^2 + \frac{1}{2} \gamma_k (\tilde{k}_k)^2 \right] + \frac{1}{2} \|\tilde{\alpha}\|^2 \right)^{1/2} \leq -\sqrt{2}\omega_k V^{1/2} \quad (41)$$



and  $\omega_k = \min\left(\sum_{k=1}^n [\lambda_k + \Omega_k / \gamma_k] 1\right)$ . So, all the error trajectories will be converged to the sliding manifold in finite time. The above inequality holds if  $\lambda_k, \Omega_k > 0$  which means that  $k_k > |W_k(t)|$  and  $\gamma_k \theta < 1$  and finally the lemma (3) yields to:

$$T_2 \leq \frac{2\left(\sum_{k=1}^n \left[\frac{1}{2} s_k(0)^2 + \frac{1}{2} (\hat{k}_k(0) - k_k)^2\right] + \frac{1}{2} \|\hat{\alpha}(0) - \alpha\|^2\right)^{1/2}}{\sqrt{2}\omega_k}.$$

### 2.3 Controller design with control singularity and input saturation

In the previous section, it has been assumed that  $g_k^{-1}(x) \neq 0$  for uncertain nonlinear second order system with Eq. (1). However, in the practical system,  $g_k(x) \neq 0$  has a feasible solution at a particular moment which leads to the control singularity. Moreover, in special applications, input saturation has been applied because of some practical and mechanical considerations. Therefore, the control input considering input saturation constraints is employed as follows:

$$u_k = \begin{cases} u_{k_{\max}} & \text{if } v_k > u_{k_{\max}} \\ v_k & \text{if } -u_{k_{\min}} < v_k < u_{k_{\max}} \\ -u_{k_{\min}} & \text{if } v_k < -u_{k_{\min}} \end{cases} \quad (42)$$

where  $v_k$  is the designed control input.  $u_{k_{\max}}$  and  $u_{k_{\min}}$  are the unknown parameters of control input saturation where  $u_{k_{\max}} \neq u_{k_{\min}}$ . To analyze the effect of control singularity and unknown non-symmetric input saturation in terms of robust tracking control, the following command is applied [30].

$$v_k = g_k(x)(g_k^2(x) + \tau_k)^{-1} v_{kr} \quad (43)$$

where  $\tau_k > 0$  is a design parameter and  $v_{kr}$  will become a new signal control in developed formation. It is clear that  $g_k(x)(g_k^2(x) + \tau_k)^{-1} = 1 - \tau_k(g_k^2(x) + \tau_k)^{-1}$ , so by substituting equations, one can obtain:

$$\begin{aligned} \dot{x}_{2k-1}(t) &= x_{2k}(t) \\ \dot{x}_{2k}(t) &= f_k(x, t) + \Delta f_k(x) + F_k(X)\alpha + g_k(x)(\Delta u_k(t) + v_k) + d_k(t) \\ &= f_k(x, t) + \Delta f_k(x) + F_k(X)\alpha + g_k(x)\Delta u_k(t) + v_{kr} - \tau_k(g_k^2(x) + \tau_k)^{-1} v_{kr} \end{aligned} \quad (44)$$

where  $\Delta u_k(t) = u_k - v_k$ .

Since the upper and the lower limit of non-symmetric input saturation are unknown and bounded,  $\Delta u_k(t)$  is unknown. For robust performance, the new disturbance with bounded limit can be considered as follows:

$$\begin{aligned} \dot{x}_{2k-1}(t) &= x_{2k}(t) \\ \dot{x}_{2k}(t) &= f_k(x, t) + \Delta f_k(x) + F_k(X)\alpha + v_{kr} + d'_k(t) \end{aligned} \quad (45)$$

where

$$d'_k(t) = g_k(x)\Delta u_k(t) - \tau_k(g_k^2(x) + \tau_k)^{-1} v_{kr} + d_k(t) \quad (46)$$

Similarly, the proposed controller could be used for system with control singularity and input saturation in the same way where  $v_{k_r}$  is the control signal which can be achieved from Eq. (21).

**Remark2.** In order to reduce the chattering effect of the sliding surface, a continuous function is adopted as [31]:

$$\text{sgn}(s_k) = \rho \frac{s_k}{|s_k| + \delta}, \quad \delta = \delta_0 + \delta_1 \|e_k\| \quad (47)$$

where  $\delta_0 > 0$  and  $\delta_1 > 0$ .

### 3.3 Ant colony optimization

Ant colony optimization is used to achieve the best tracking performance. Since the ACO algorithm is very suitable for parallel working, the sliding surface constants could be tuned by the ACO in suitable ranges. Every two states of second order dynamic systems are controlled with one sliding surface which possesses three constant gains: therefore the number of parameters of the controller will be determined up to their degrees of freedom. To optimize the controller design with ACO, all of the values for each parameter are placed in different vectors. In order to create a graph representation of the problem, these vectors are considered as paths between nests.

Each ant is placed at different or same corners, at the beginning of the problem. Eq. (51) (probability equation) determines these ants will situate on which adjacent node at time (t) [32].

$$P_{ij}^k(t) = \begin{cases} \frac{[\tau_{ij}(t)]^\sigma [\eta_{ij}(t)]^\beta}{\sum_{i \in N_i} [\tau_{ij}(t)]^\sigma [\eta_{ij}(t)]^\beta} & \text{if the } k \text{ is an allowed selection} \\ 0 & \text{if otherwise} \end{cases} \quad (48)$$

where  $\tau_{ij}(t)$  is the trace amount of pheromones at  $(i, j)$  at time  $t$ ,  $\eta_{ij}(t)$  is the visibility value between  $(i, j)$ ,  $\sigma$  shows the relative importance of the pheromone trace in the problem,  $\beta$  denotes the importance given to the visibility value and  $N_i$  is a set of the node that hasn't been chosen yet. The amount of pheromone trace is updated according to the following equation [33]

$$\tau_{ij}(t+n) = (1-p)\tau_{ij}(t) + \Delta\tau_{ij}(t) \quad (49)$$

where  $p$  is the proportion of pheromone trace evaporated between  $t$  and  $t+1$  time period ( $0 < p < 1$ ) and  $\Delta\tau_{ij}(t)$  shows the amount of pheromone trace of the corner due to the election of the  $(i, j)$  during a tour of the ant. This value is reached from below:

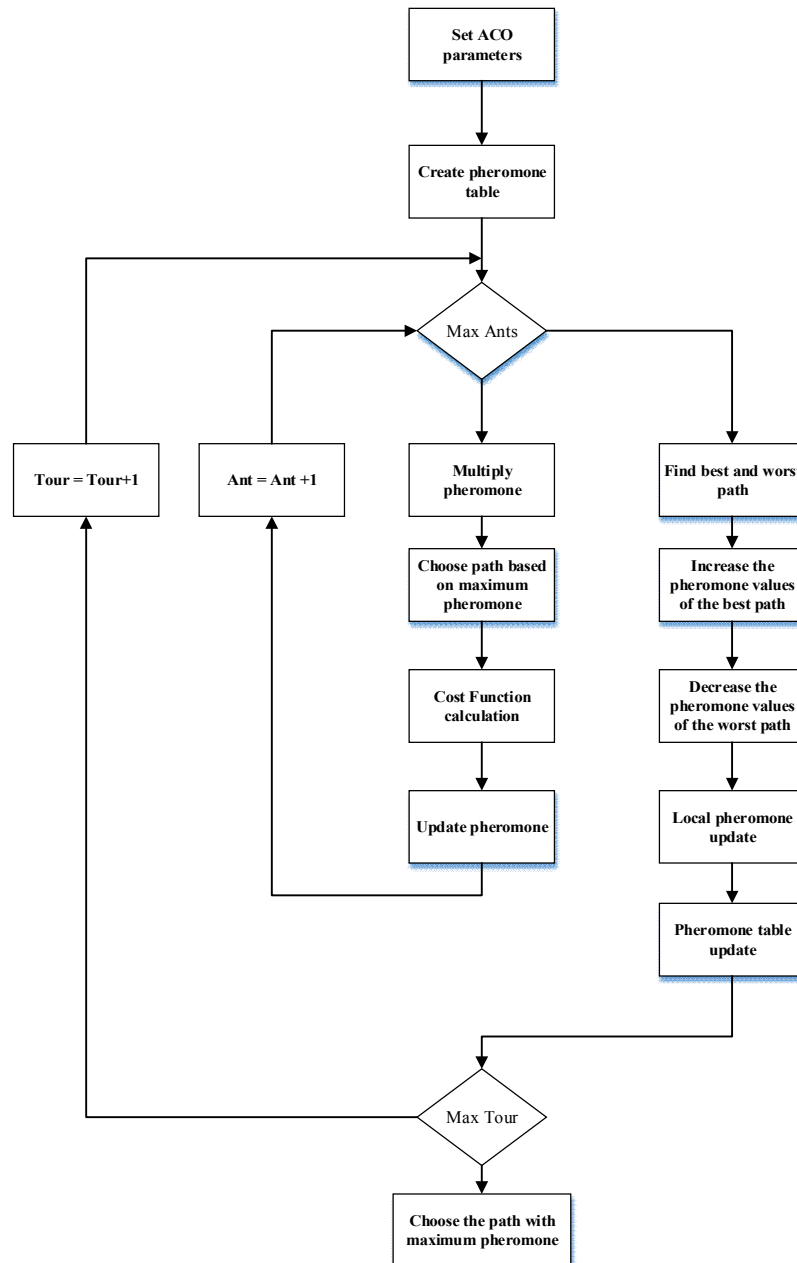
$$\Delta\tau_{ij} = \sum_{k=1}^m \Delta\tau_{ij}^k \quad (50)$$

$m$  is the total ant number,  $\Delta\tau_{ij}^k$  is the amount of pheromone trace left by  $k^{\text{th}}$  ant at  $(i, j)$  and computed from [26]:

$$\Delta\tau_{ij}^k = \frac{Q}{L_k} \quad (51)$$

$Q$  is a constant and  $L_k$  is the tour length of  $k^{th}$  ant [26].

It should be noted that the parameters of sliding surface have been chosen such that the conditions of Lemma (3) are satisfied. To solve this problem, the interval ranges of these parameters for ACO should be selected correctly. Furthermore, the flowchart of ACO algorithm has been introduced in Fig 1.



**Figure 1. The flowchart of the ACO algorithm.**

Some of the important advantages of ACO algorithm can be summarized as follows:

- It has advantage of distributed computing.
- It is robust and also easy to match with other algorithms.
- When the values of graph changes drastically, the ant colony algorithm can be run continuously and adapt to changes in real time.

Finally, a block diagram of the proposed control has been brought to show the overall scheme of the designed controller.

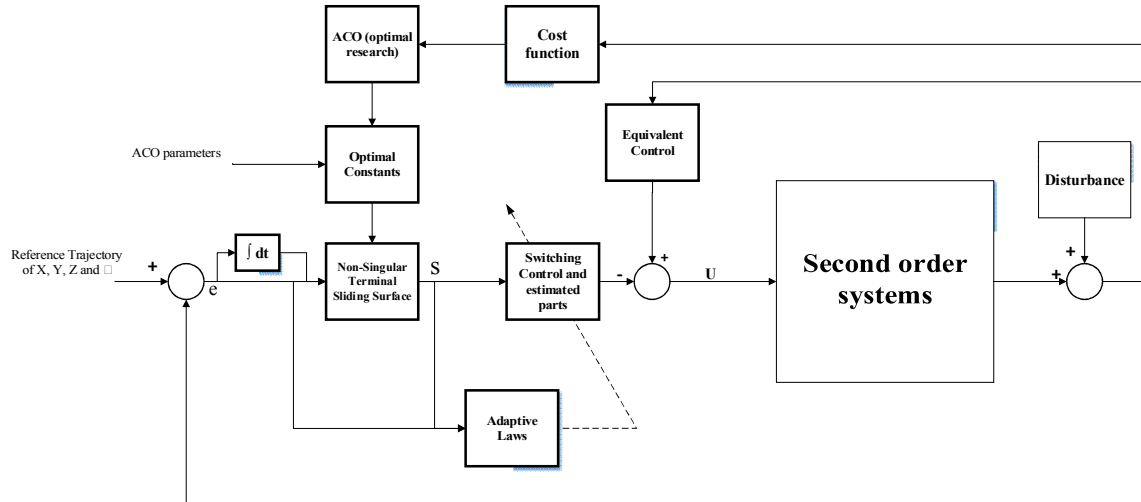


Figure 2. A flowchart of the proposed algorithm.

#### 4 Case study

The two-degree of freedom (2DOF) manipulator has been chosen as case study to demonstrate the effectiveness of the proposed controller. The dynamic equations of a 2DOF polar robot manipulator are stated as below, considering external disturbances and unknown parameters (Faieghi et al., 2012):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{[\mu x_1 + M(x_1 + a)]x_4^2 + u_1 + d_1}{(\mu + M)} + (x_1 - x_2)\alpha \end{aligned} \quad (52)$$

$$\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{-2[\mu x_1 + M(x_1 + a)]x_2 x_4 + u_2 + d_2}{J_1 + J_2 + \mu x_1^2 + M(x_1 + a)^2} + x_4\alpha_2 \end{aligned}$$

$X = [x_1, x_2, x_3, x_4]$  is the state vector, where  $x_1$  is the position of the center arm,  $x_2$  is the center arm speed,  $x_3$  is the angular position of the arm,  $x_4$  is the angular velocity of the arm,  $\mu$  is the mass of motional link,  $M$  is the payload,  $J_1$  and  $J_2$  are the moments of inertia of the motional link with respect to the vertical axis.  $d_1(t)$  and  $d_2(t)$  are unknown but bounded external disturbances. All the noted assumptions are considered in the following simulations and these parameters are different with similar parameters in the controller.

## 5 Simulation

In this section, simulation results are given by MATLAB/SIMULINK software and ODE45 solver, to illustrate the effectiveness of the proposed controller in the presence of external disturbances and unknown parameters. The external disturbances are chosen as  $d_1(t) = 0.3 \cos(4\pi t)$  and  $d_2(t) = 0.5 \cos(4\pi t)$ . The unknown parameters are chosen as  $\alpha_1 = 1$  and  $\alpha_2 = 1$ , where D in Eq. (11) are equal to 5 and 8. The parameters of the robot manipulator are set as:  $M = 1.5$  kg,  $\mu = 1$  kg,  $J_1 = J_2 = 1$  Kg m<sup>2</sup> and  $a = 1$  m. The initial conditions are chosen as the following vector:  $x(0) = [-0.2, -0.25, 0.98, 0.36]^T$ . The desired trajectories are:

$$x_{1r}(t) = 0.2 \cos(0.5t) \text{ m} \quad (53)$$

$$x_{3r}(t) = 0.2 \cos(0.5t) \text{ rad} \quad (54)$$

The parameters of ACO algorithm are set as follows:

Number of ants = 10; Pheromone = 0.06; Evaporation parameter = 0.95; Positive Pheromone = 0.2; Negative pheromone = 0.3; and Maximum tour = 10.

Population based algorithm which have been introduced, need offline operation to optimize the different problems according to the cost function. Therefore, it is possible to gather a set of valuable training data around their paths of tracking or their equilibrium or even around their operating points in linear and nonlinear systems. A set of training data in the defined path of the robot for the special period of time has been considered as an input-output data in this paper. The proposed cost function has been chosen as following in order to increase the accuracy of tracking:

$$C = \sum_{i=1}^k e_i^2 \quad (55)$$

The gains of each sliding surface should be chosen as conditions of lemma (3). Therefore, Hurwitz stabilization condition leads to following inequality:

$$k \in [k_{\min}, k_{\max}], \quad k_k^p > \frac{k_k^i}{k_k^d} \quad (56)$$

So, the following inequality is resulted:

$$k_k^p(\min) > \frac{k_k^i(\max)}{k_k^d(\min)} \quad (57)$$

where each interval is chosen as  $k_1^p \in [10, 16]$ ,  $k_1^i \in [0, 5]$ ,  $k_1^d \in [5.9]$  and  $k_2^p \in [20, 25]$ ,  $k_2^i \in [8, 12]$ ,  $k_2^d \in [6, 10]$ .

So the parameters of the controller are optimized as  $k_1^p = 10$ ,  $k_1^i = 1.1480$ ,  $k_1^d = 8.9760$  and  $k_2^p = 21.4100$ ,  $k_2^i = 9.8480$ ,  $k_2^d = 9.2160$ .

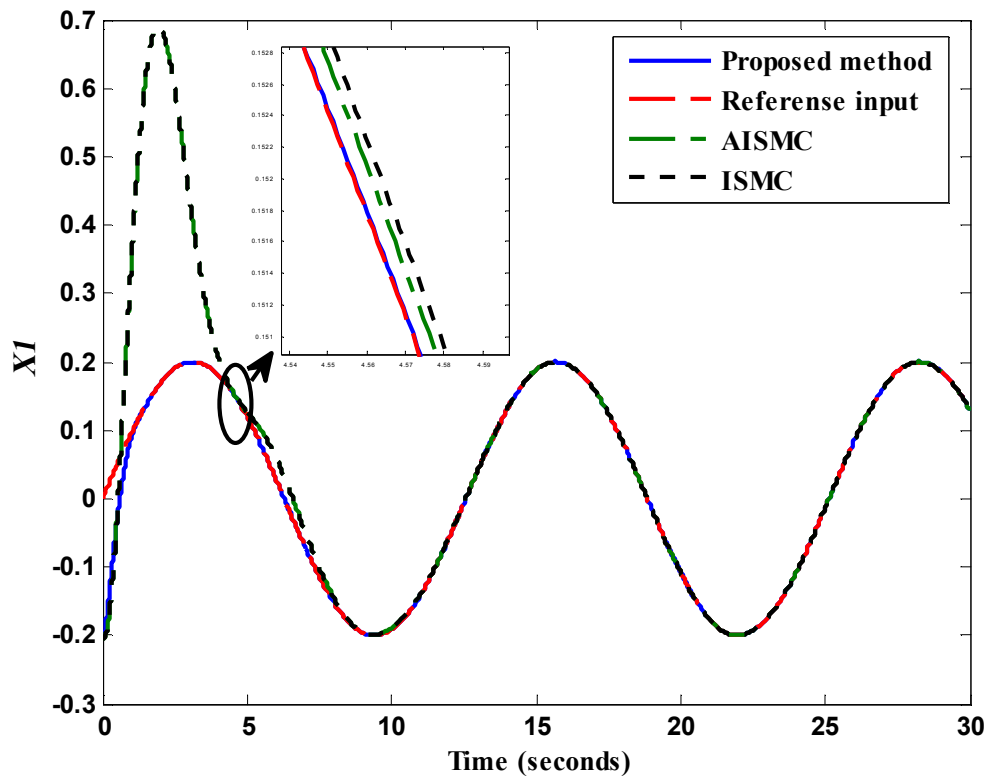
The parameters of adaptation law are chosen  $\gamma = 1$  and  $\theta = 0.5$  for all values of  $k$ .

$\delta_0 = 0.001$  and  $\delta_1 = 0$  are set for alternative sign function.

The parameters of the control input saturation are  $\tau_1 = 0.001$ ,  $\tau_2 = 0.01$ ,  $u_{1\max} = 70$ ,  $u_{1\min} = -10$  and  $u_{2\max} = 50$ ,  $u_{2\min} = -20$ .

Figs 3 and 4 show the time response of systems output in the presence of disturbance and unknown parameters. In these figures, a comparison study has been done to show the effectiveness of the proposed controller against conventional adaptive integral

sliding mode (AISMC) and integral sliding mode (ISMC [2]) methods. AISMC and ISMC use linear surfaces. As it can be seen, steady state errors and transient responses have been improved with using proposed method. Remark 2 has been used to decrease chattering effect that usually occurs in control input (Figures 7, 8). Figs 5, 6 show errors of state Eq. (13) where they convergence to zero in finite time. Augmented integral states guarantee zero steady-state error. Figs 9, 10 show the estimation performance of the adaptive gains, they have converged to their equilibrium from initial values. Sliding surface time responses are also shown in Figs 11, 12.



*Figure 3. Tracking response of joint 1 without input saturation*

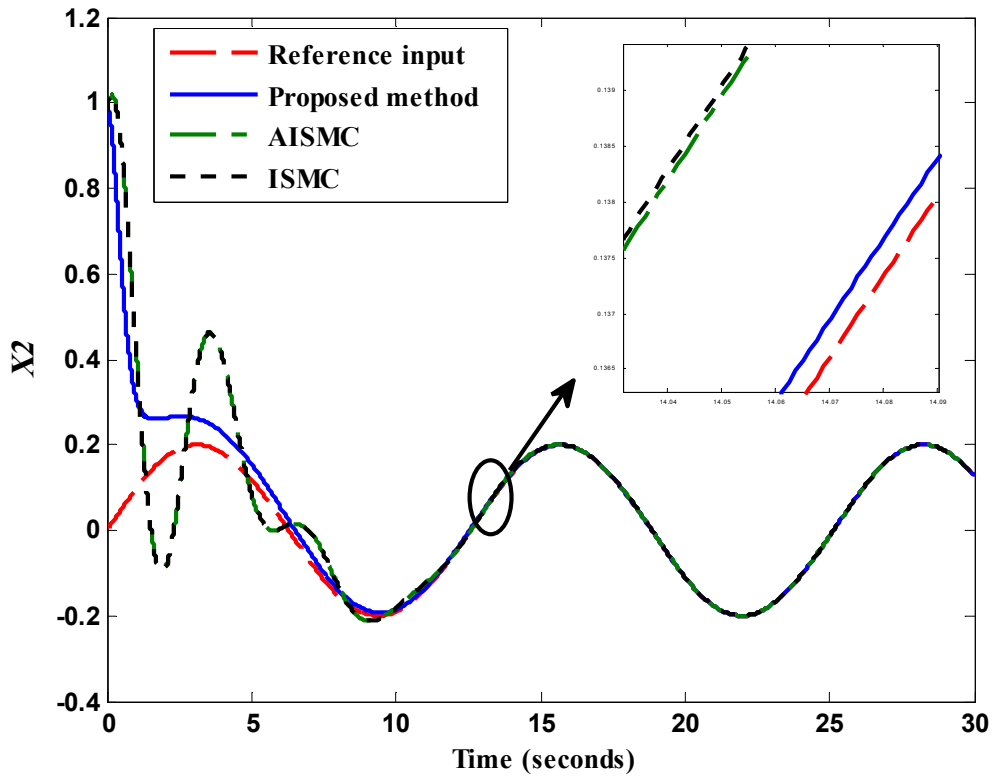


Figure 4. Tracking response of joint 2 without input saturation

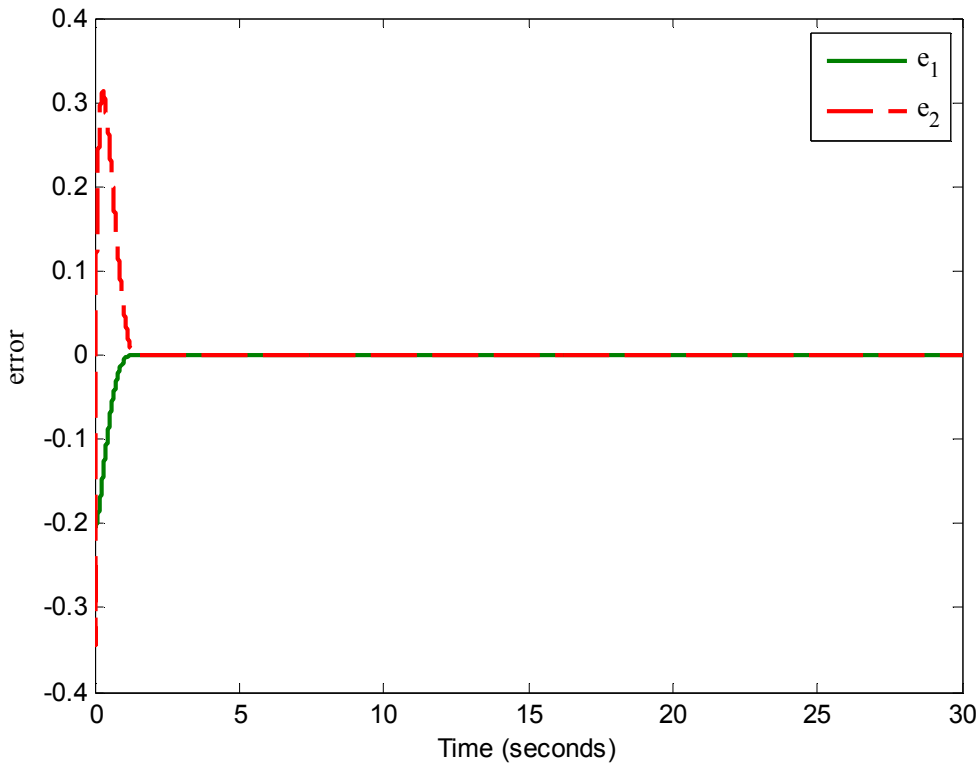


Figure 5. Response errors of joint 1.

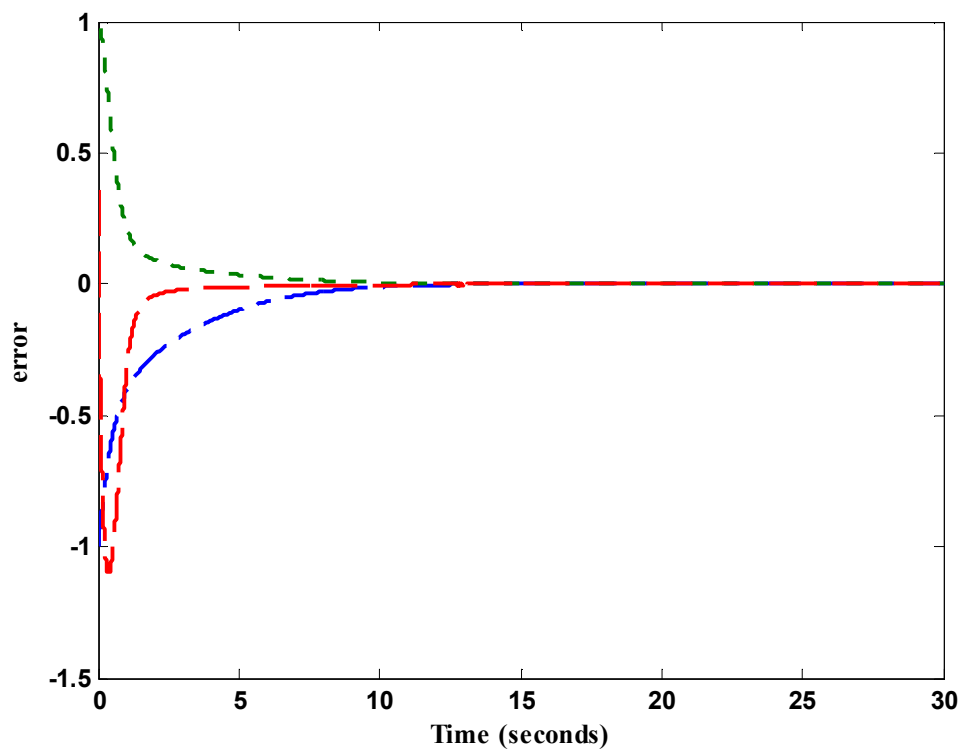


Figure 6. Response errors of joint 2 :  $\zeta_3$  (dash-dot line),  $e_3$  (dot) and  $e_4$  (dash).

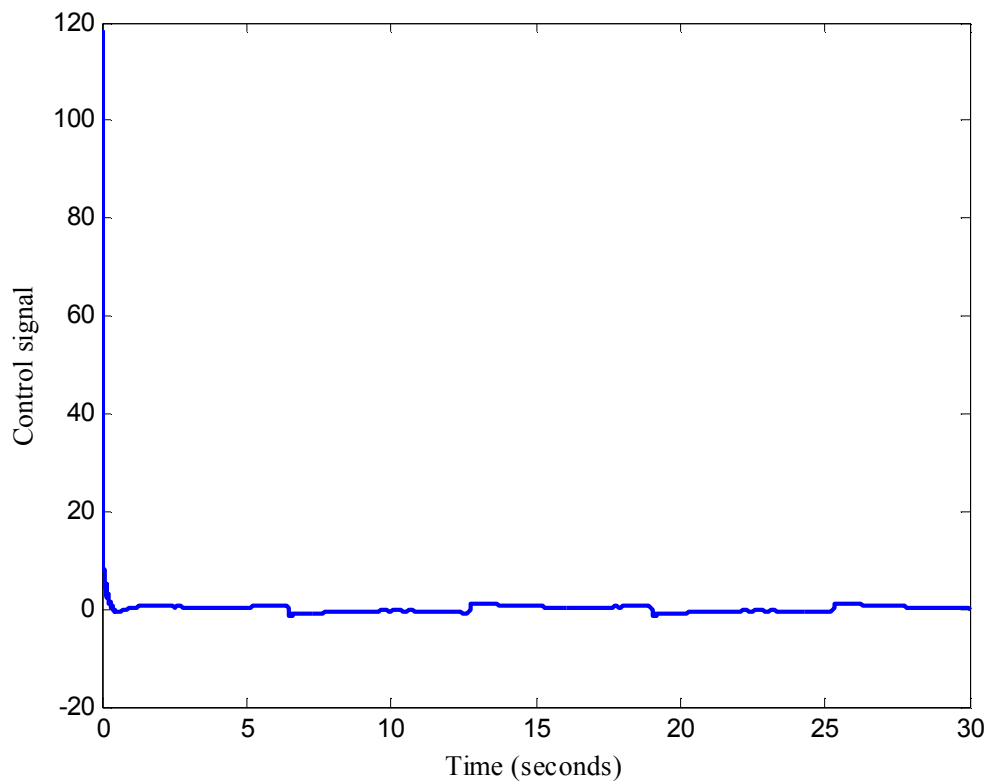


Figure 7. Control input  $u_1(t)$  without input saturation



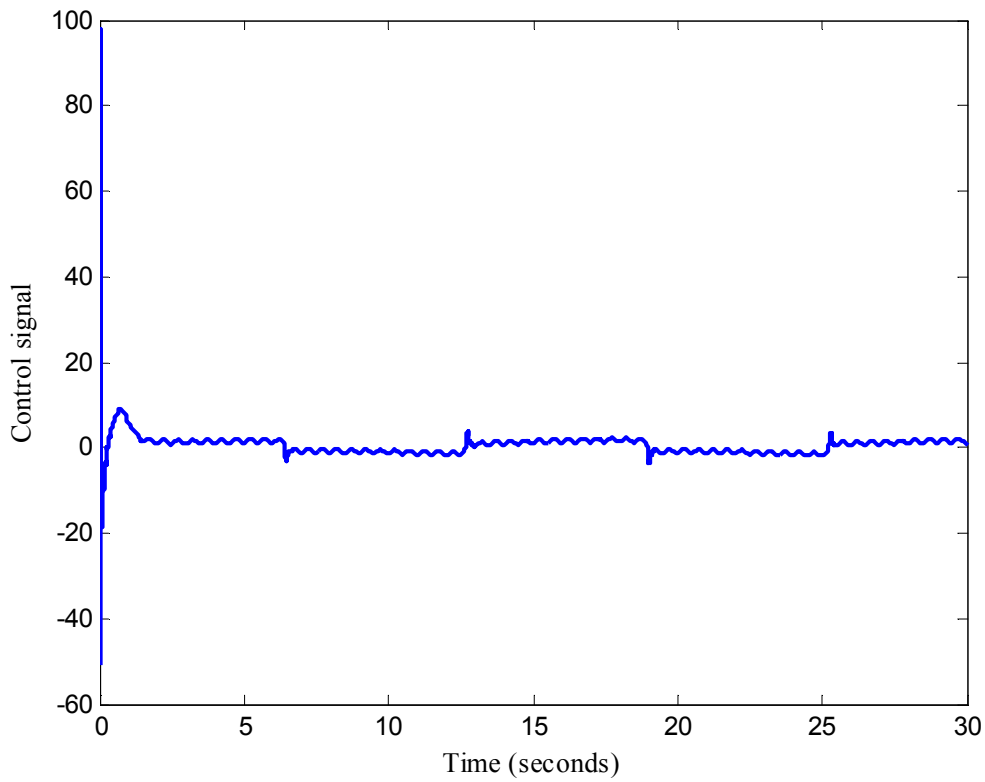


Figure 8. Control input  $w_2(t)$  without input saturation

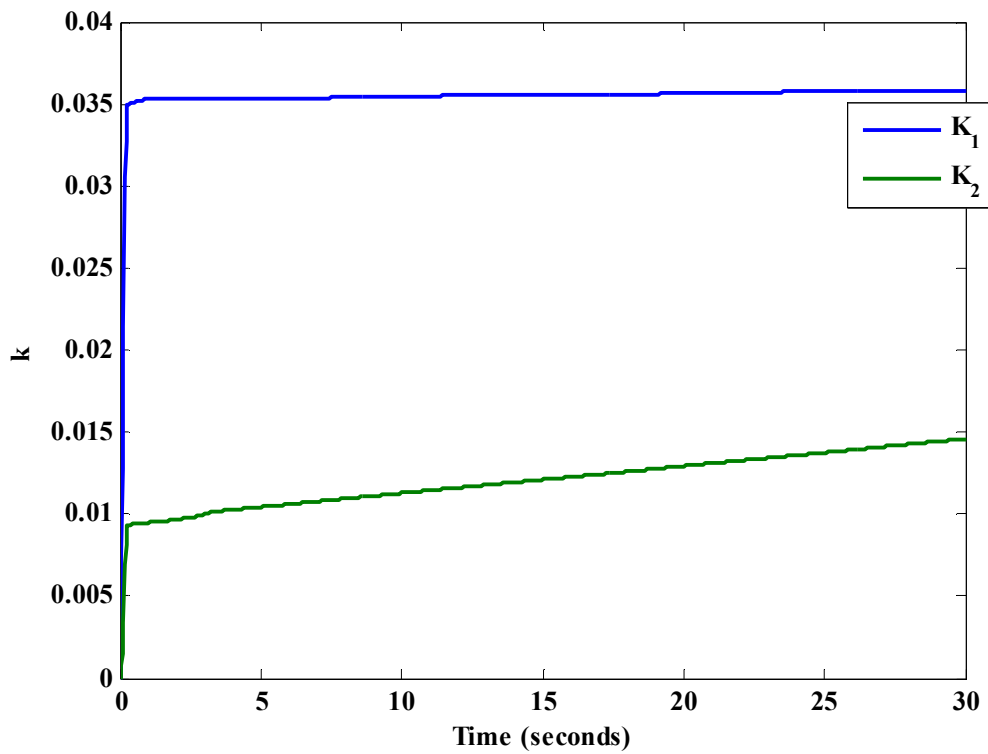


Figure 9. Time response of adaptation parameters without input saturation

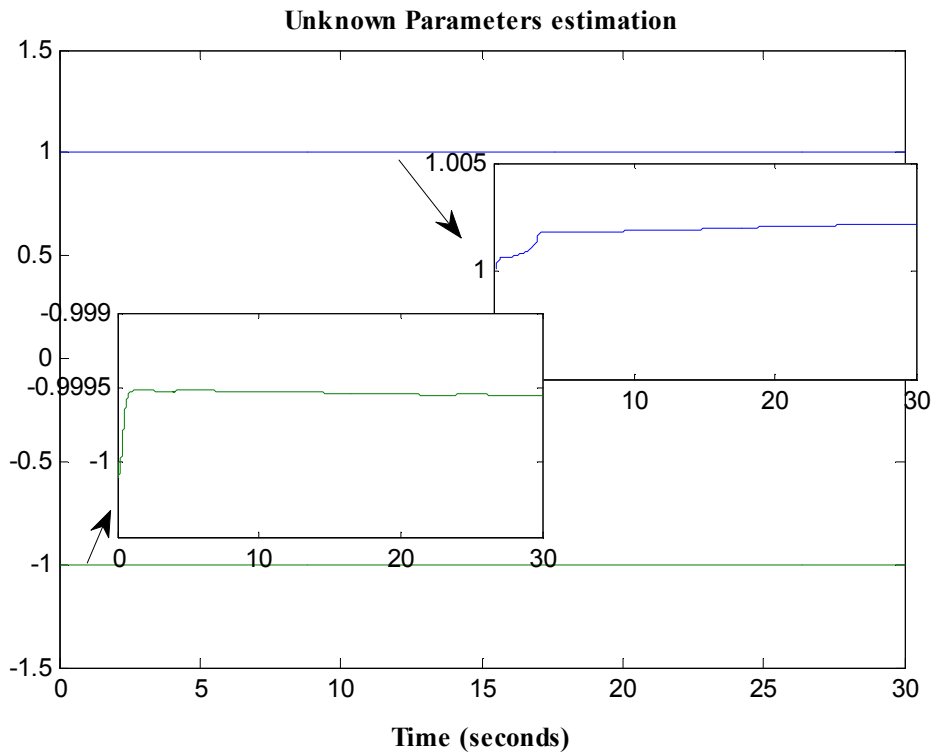


Figure 10. Unknown parameters estimation,  $\hat{\alpha}_1$  (blue) and  $\hat{\alpha}_2$  (green)

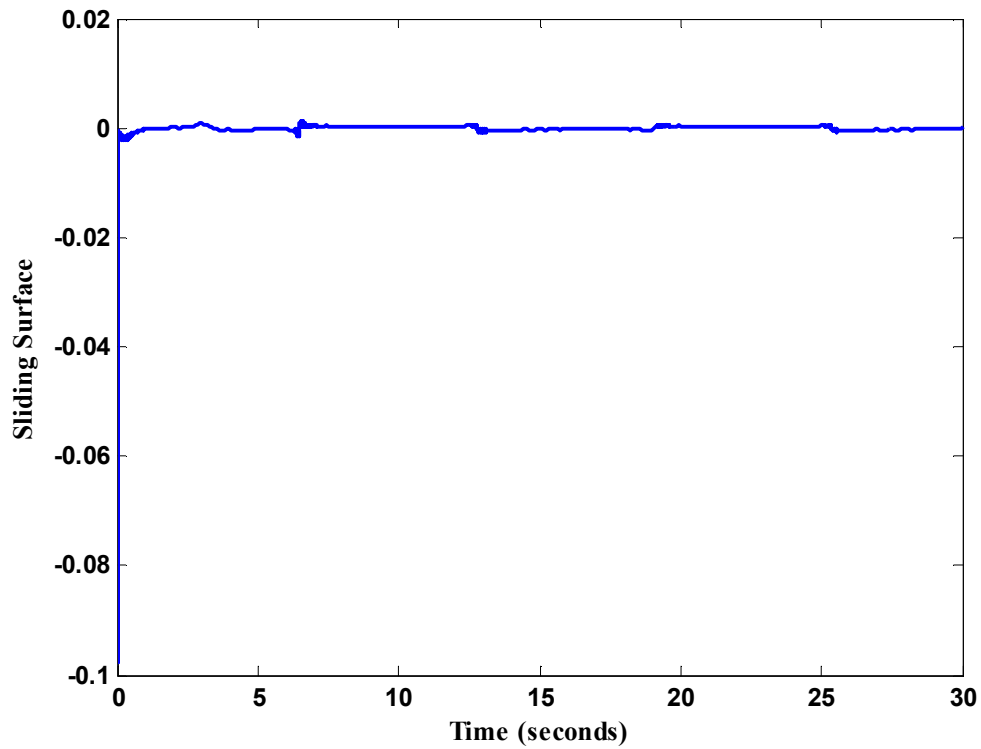
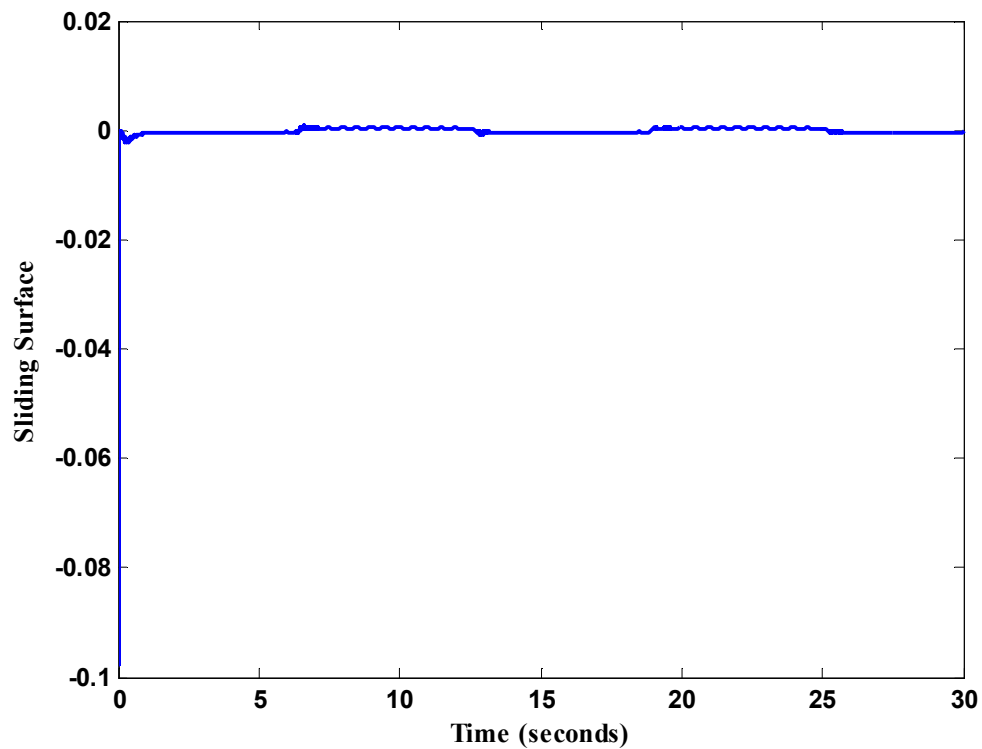
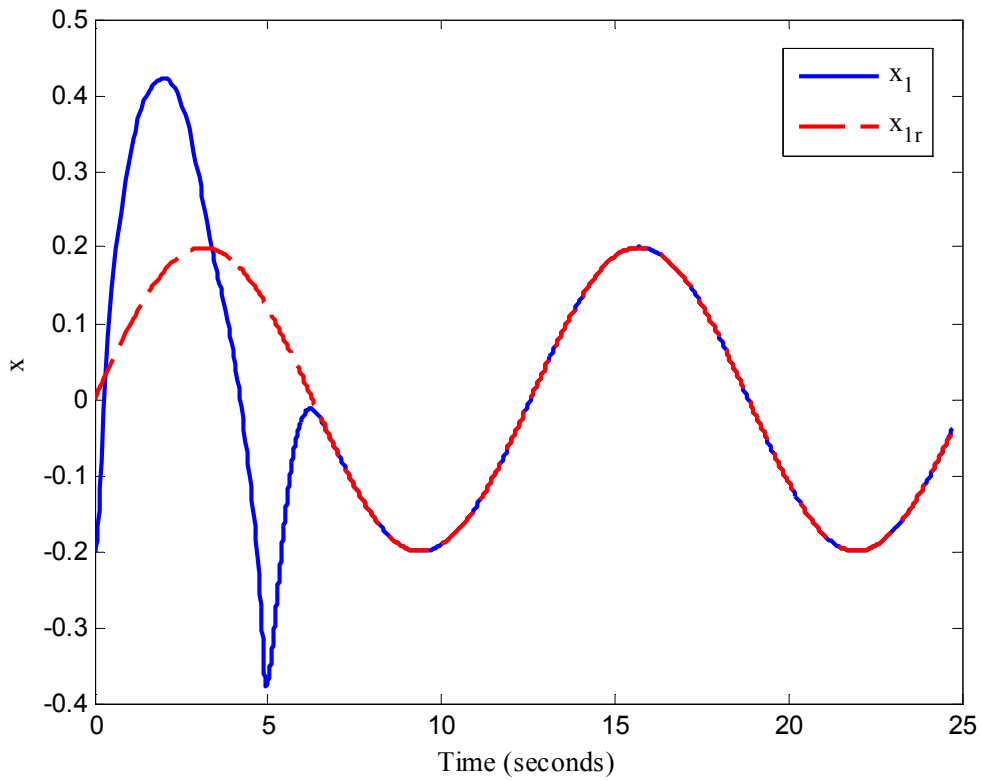


Figure 11. Sliding surface for joint 1 without input saturation

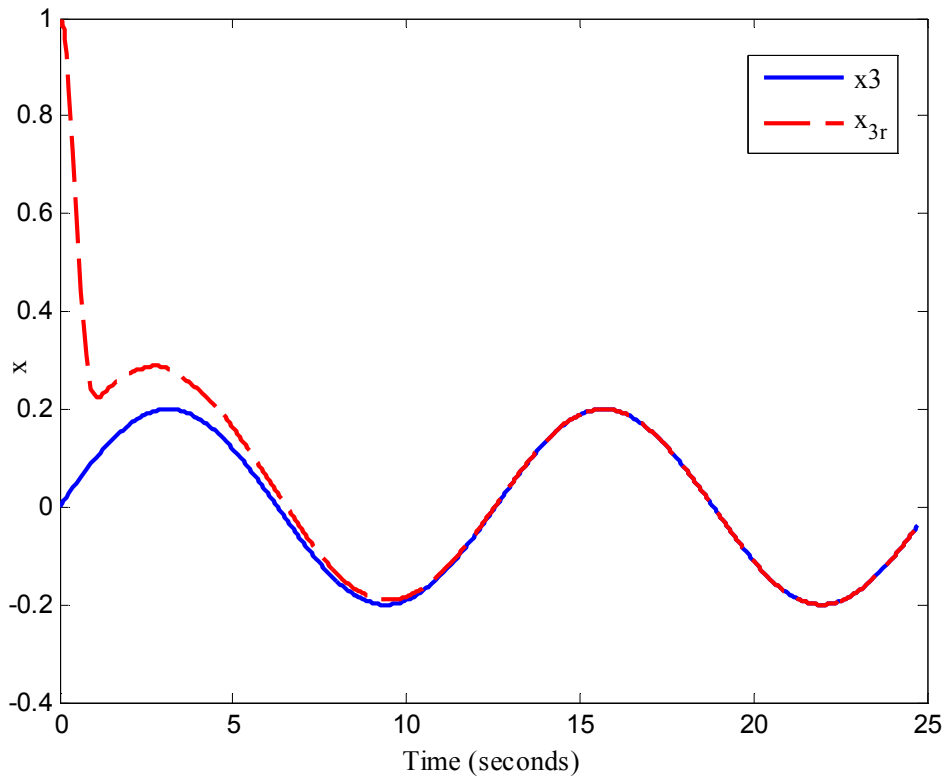


*Figure 12. Sliding surface for joint 2 without input saturation*

The following figures display the controller performance in the presence of input saturation. Figs 15, 16 show the saturation effects in the control input. It can be seen that whenever the control signals have to be bounded by saturation action, the signal reshape to compensate this effects.



**Figure 13. Tracking response of joint 1 with input saturation**



**Figure 14. Tracking response of joint 2 with input saturation**

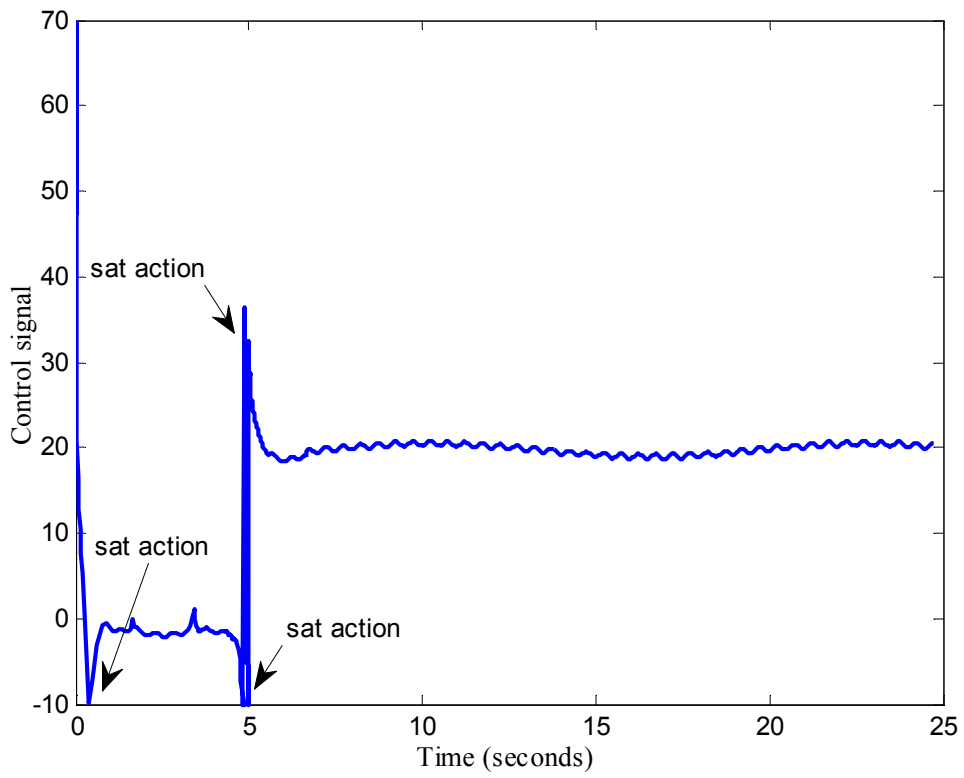


Figure 15. Control input  $v_1(t)$  with input saturation

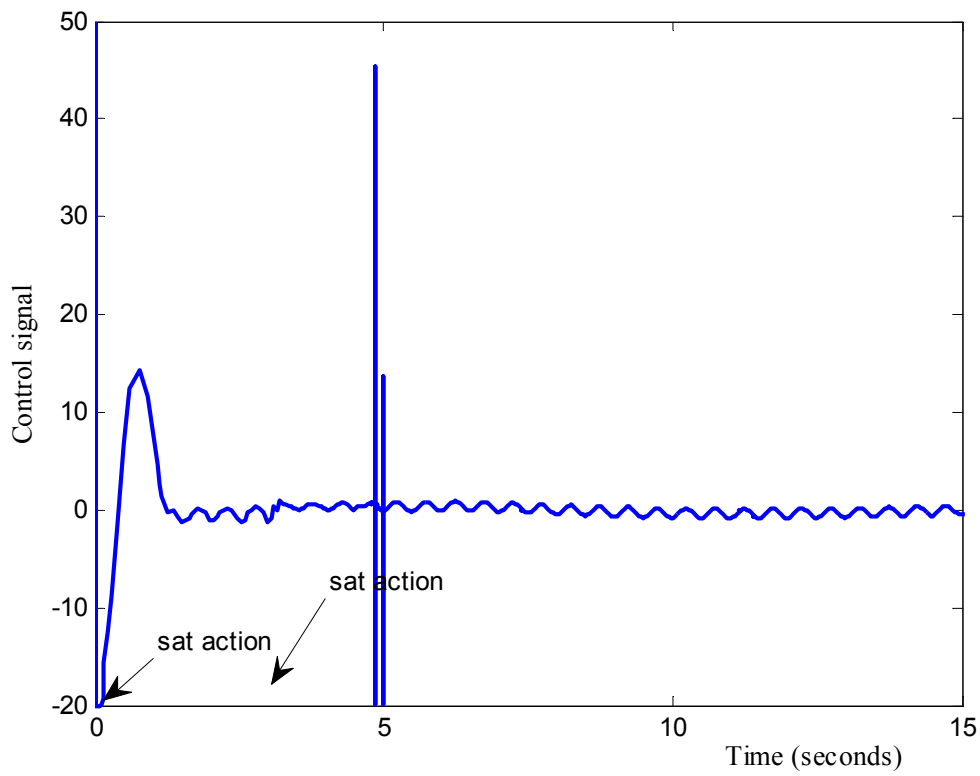


Figure 16. Control input  $v_2(t)$  with input saturation

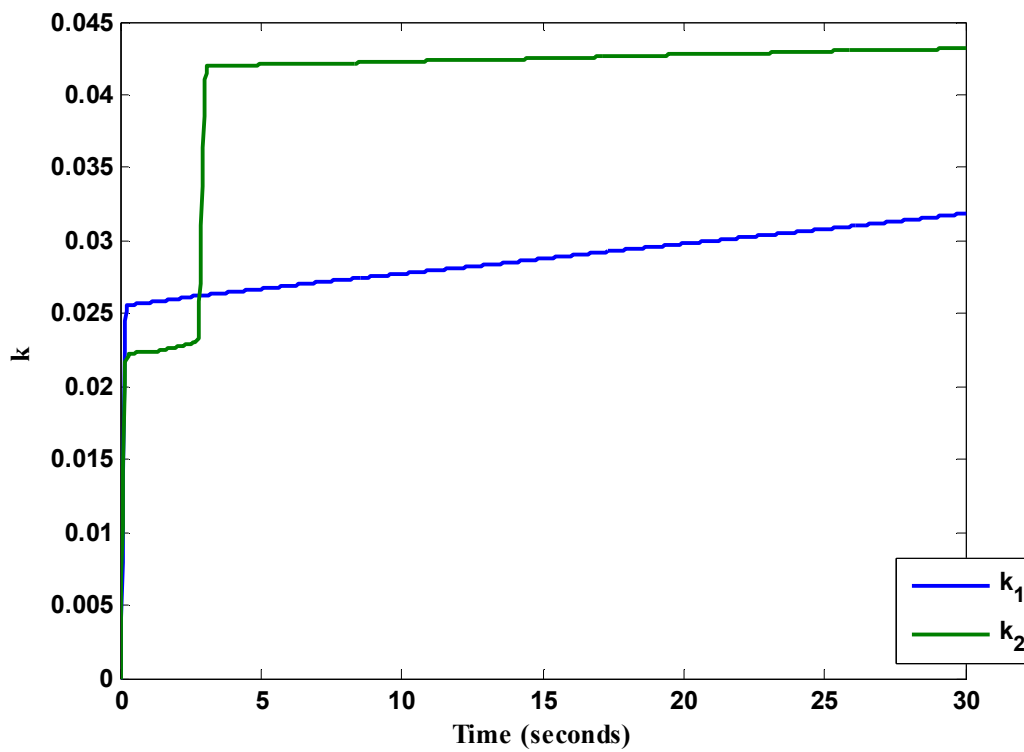
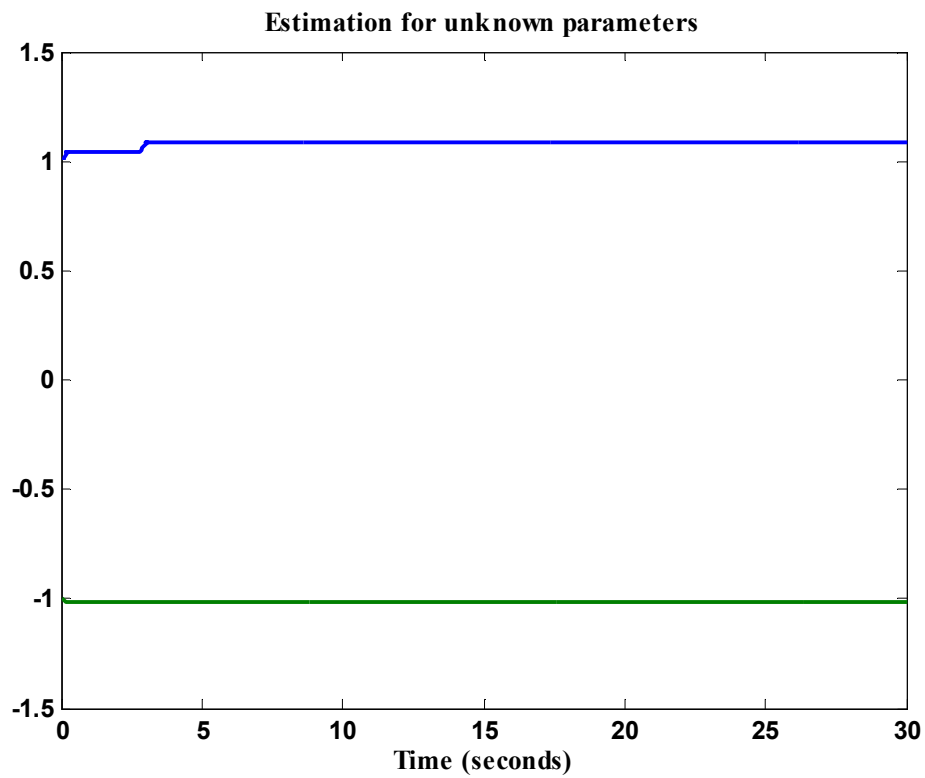


Figure 17. Time response of adaptation parameters with input saturation



**Figure 18.** Time response of adaptive vector parameters  $\hat{a}_1$  (blue) and  $\hat{a}_2$  (green) with input saturation.

The controller performance has been shown for another reference inputs in Fig 19. The tracking has been satisfied appropriately despite of defined disturbance and uncertainty.

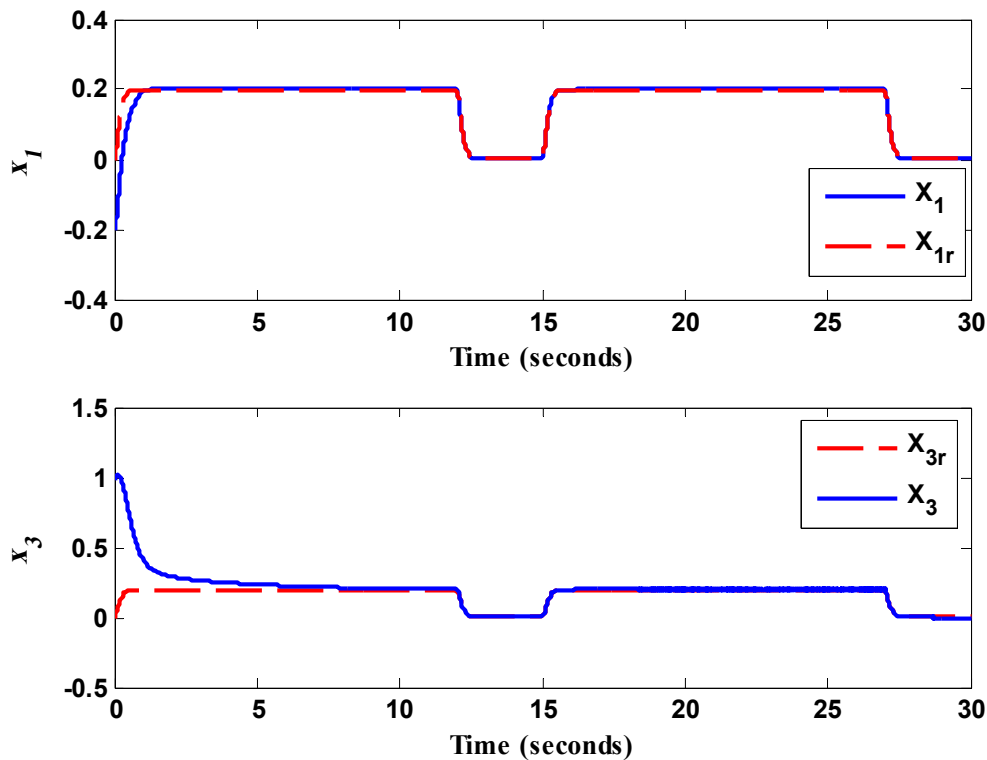


Figure 19. Time response of reference input signals and joint 1 and 2 in the presence of uncertainty, unknown parameters and disturbance.

## 6 Conclusions

In this paper, the tracking control of second order systems has been developed in the presence of uncertainties, disturbances, unknown parameters and input saturation. A robust adaptive integral sliding mode control based on finite time homogeneous approach has been proposed for second order systems with perturbations. Adaptive tuning law has been designed to deal with unknown parameters and the unknown bounded system uncertainty or disturbance. The upper bound of uncertainty and disturbances are not required to be known. Augmented integral state has been added to the controller for zero steady-state errors and robustness.

Non-symmetric input saturation has been introduced to deal with singularity which may occur in input control or in practical applications. Since, choosing desired constant parameters of the controller is difficult, ant colony optimization has been used to deal with this problem in offline tests among specified intervals. The stability of the controlled system is proved using Lyapunov stability criterion. Simulation results demonstrate the advantages and performance of the proposed strategy.



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